KTH Teknikvetenskap

SF2729 GROUPS AND RINGS
HOMEWORK ASSIGNMENT II
RINGS

The following homework problems can count as the first problem in the second section of the final exam. The solutions should be handed in no later than on April 20. The computations and arguments should be easy to follow. Collaborations should be clearly stated.

Credits on homework	31-35	26-30	21-25	16-20	11-15	6-10	0-5
Credits on problem 1 of part II	6	5	4	3	2	1	0

Problem 1. Let R be a ring and let $ev_r : R[x] \to R$ be the evaluation map. Show that R is a commutative ring if and only if ev_r is a ring-homomorphism for every $r \in R$. **(4)**

Problem 2. (1) Let R_1, R_2 be two rings with unity. Let R^* denote the group of invertible elements. Show that $R_1^* \times R_2^* \cong (R_1 \times R_2)^*$ (isomorphic as groups). **(3)**

(2) Use the previous part to show that if $n, m \in \mathbb{Z}$ are relatively prime then

$$\phi(n)\phi(m) = \phi(nm),$$

where ϕ denotes the Euler function. **(3)**

Problem 3. (1) Let R be a ring. Define operations on $\mathbb{Z} \times R$ so that $\mathbb{Z} \times R$ is a ring with a unity. **(3)**

(2) Show that every ring R can be embedded[1] in the ring of endomorphisms of an abelian group. **(5)**

Problem 4. Let R be a commutative ring with unity, $a \in R$ and let $I \subseteq R$ be an ideal. Consider:

$$\mathcal{I} = \{f(x) \in R[x] \text{ such that } f(a) \in I\}.$$

(1) Show that $\mathcal{I}$ is an ideal of $R[x]$. **(2)**
(2) Show that $\mathcal{I}$ is prime if and only if I is prime. **(4)**
(3) Identify $\mathcal{I}$ for $R = \mathbb{Z}, a = 1$ and $I = (5)$. **(4)**

Problem 5. Let p be a prime number. Identify $\mathbf{Z}_{p^r}^*$, the group of units of the ring $\mathbf{Z}_{p^r}$ (note that the cases $p = 2$ and $p > 2$ are different). **(7)**

[1] A ring R is embedded in a ring S if there is a ring homomorphism $S \to R$ which is injective.